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# Stochastic Hopfield neural networks

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**Abstract**

Hopfield (1984 *Proc. Natl Acad. Sci. USA* **81** 3088–92) showed that the time evolution of a symmetric neural network is a motion in state space that seeks out minima in the system energy (i.e. the limit set of the system). In practice, a neural network is often subject to environmental noise. It is therefore useful and interesting to find out whether the system still approaches some limit set under stochastic perturbation. In this paper, we will give a number of useful bounds for the noise intensity under which the stochastic neural network will approach its limit set.

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## 1. Introduction

Much of the current interest in artificial networks stems not only from their richness as a theoretical model of collective dynamics but also from the promise they have shown as a practical tool for performing parallel computation (cf Denker [3]). Theoretical understanding of neural-network dynamics has advanced greatly in the past fifteen years (cf [2, 5–7, 9, 14]). The neural network proposed by Hopfield [5] can be described by an ordinary differential equation of the form

$$C_i\dot{x}_i(t) = -\frac{1}{R_i}x_i(t) + \sum_{j=1}^{n} T_{ij}g_j(x_j(t)) + I_i \qquad 1 \leqslant i \leqslant n \tag{1.1}$$

on $t \geqslant 0$. The variable $x_i(t)$ represents the voltage on the input of the $i$th neuron, and $I_i$ is the external input current to the $i$th neuron. Each neuron is characterized by an input capacitance $C_i$ and a transfer function $g_i(u)$. The connection matrix element $T_{ij}$ has a value $+1/R_{ij}$ when the noninverting output of the $j$th neuron is connected to the input of the $i$th neuron through a

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resistance $R_{ij}$, and a value $-1/R_{ij}$ when the inverting output of the $j$th neuron is connected to the input of the $i$th neuron through a resistance $R_{ij}$. The parallel resistance at the input of each neuron is defined by $R_i = \left(\sum_{j=1}^{n} |T_{ij}|\right)^{-1}$. The nonlinear transfer function $g_i(u)$ is sigmoidal, saturating at $\pm 1$ with maximum slope at $u = 0$. By defining

$$b_i = \frac{1}{C_i R_i} \qquad a_{ij} = \frac{T_{ij}}{C_i} \qquad c_i = \frac{I_i}{C_i}$$

equation (1.1) can be re-written as

$$\dot{x}_i(t) = -b_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + c_i \qquad 1 \leqslant i \leqslant n;$$

that is, in the matrix form,

$$\dot{x}(t) = -Bx(t) + Ag(x(t)) + C \tag{1.2}$$

where

$$x(t) = (x_1(t), \ldots, x_n(t))^T \qquad B = \mathrm{diag}(b_1, \ldots, b_n) \qquad A = (a_{ij})_{n\times n}$$
$$C = (c_1, \ldots, c_n)^T \qquad g(x) = (g_1(x_1), \ldots, g_n(x_n))^T.$$

Note that we always have

$$b_i = \sum_{j=1}^{n} |a_{ij}| > 0 \qquad \text{and} \qquad c_i \geqslant 0 \qquad 1 \leqslant i \leqslant n. \tag{1.3}$$

Moreover, we assume in this paper that the network is symmetric in the sense

$$a_{ij} = a_{ji} \qquad 1 \leqslant i \quad j \leqslant n \tag{1.4}$$

that is, $A$ is a symmetric matrix.

We mentioned above that the nonlinear transfer function $g_i(u)$ is sigmoidal, saturating at $\pm 1$ with maximum slope at $u = 0$. To be more precise, let us state the properties of $g_i$ below:

- $g_i(u)$ is strictly increasing, $-1 < g_i(u) < 1$ and $g_i(0) = 0$;
- $\dot{g}_i(u) := \frac{\mathrm{d}}{\mathrm{d}u} g_i(u) > 0$, it increases on $u < 0$, reaches its maximum $\beta_i := \dot{g}_i(0)$ at $u = 0$ and then decreases on $u > 0$;
- $\ddot{g}_i(u) := \frac{\mathrm{d}^2}{\mathrm{d}u^2} g_i(u)$ is bounded, $\ddot{g}_i(u) > 0$ for $u < 0$, $\ddot{g}_i(0) = 0$ and $\ddot{g}_i(u) < 0$ for $u > 0$.

Moreover, $g_i(u)$ approaches its asymptotes $\pm 1$ very slowly such that

$$\int_0^{\pm\infty} u \dot{g}_i(u)\, \mathrm{d}u = \infty. \tag{1.5}$$

Let us now define a $C^2$-function $V : R^n \to R$ by

$$V(x) = \sum_{i=1}^{n} b_i \int_0^{x_i} u \dot{g}_i(u)\, \mathrm{d}u - \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} g_i(x_i) g_j(x_j) - \sum_{i=1}^{n} c_i g_i(x_i). \tag{1.6}$$

Let $x(t)$ be a solution to network (1.2). It is easy to compute

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} = -\sum_{i=1}^{n} \dot{g}_i(x_i) \left( -b_i x_i + \sum_{j=1}^{n} a_{ij} g_j(x_j) + c_i \right)^2.$$

Recalling the fact that $\dot{g}_i(x_i) > 0$, we see that

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} < 0$$

unless $-b_i x_i + \sum_{j=1}^n a_{ij} g_j(x_j) + c_i = 0$ for all $1 \leqslant i \leqslant n$, where

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} = 0.$$

It is due to this nonpositive property of $\mathrm{d}V(x(t))/\mathrm{d}t$ that Hopfield [6] shows that the time evolution of the system is a motion in state space that seeks out minima in the system energy. More precisely, the solution will approach the set

$$K_0 := \left\{ x \in R^n : -b_i x_i + \sum_{j=1}^n a_{ij} g_j(x_j) + c_i = 0, 1 \leqslant i \leqslant n \right\}. \tag{1.7}$$

However, a neural network is often subject to environmental noise. For example, if every external input $I_i$ is perturbed in the way $I_i \to I_i + \varepsilon_i \dot{w}_1(t)$, where $\dot{w}_1(t)$ is a white noise, then the stochastically perturbed neural network is described by a stochastic differential equation

$$\mathrm{d}x(t) = [-Bx(t) + Ag(x(t)) + C]\,\mathrm{d}t + \sigma_1\,\mathrm{d}w_1(t) \tag{1.8}$$

where $\sigma_1 = (\varepsilon_1/C_1, \ldots, \varepsilon_n/C_n)^T$. If, moreover, the connection matrix element $T_{ij}$ is perturbed in the way $T_{ij} \to T_{ij} + \varepsilon_{ij}\dot{w}_2(t)$, where $\dot{w}_2(t)$ is another white noise independent of $\dot{w}_1(t)$, then the stochastically perturbed neural network can be described as

$$\mathrm{d}x(t) = [-Bx(t) + Ag(x(t)) + C]\,\mathrm{d}t + \sigma_1\,\mathrm{d}w_1(t) + \sigma_2 g(x(t))\,\mathrm{d}w_2(t) \tag{1.9}$$

where $\sigma_2 = (\varepsilon_{ij}/C_i)_{n\times n}$. In general, we may describe the stochastic neural network by a stochastic differential equation

$$\mathrm{d}x(t) = [-Bx(t) + Ag(x(t)) + C]\,\mathrm{d}t + \sigma(x(t))\,\mathrm{d}w(t) \tag{1.10}$$

on $t \geqslant 0$. Here $w(t) = (w_1(t), \ldots, w_m(t))^T$ is an $m$-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}\}_{t\geqslant 0}$ (i.e. $\mathcal{F}_t = \sigma\{w(s) : 0 \leqslant s \leqslant t\}$), and $\sigma : R^n \to R^{n\times m}$, i.e. $\sigma(x) = (\sigma_{ij}(x))_{n\times m}$ which is called the *noise intensity matrix*. The question is: does the solution of the network under stochastic perturbation still approach $K_0$ or a different limit set? The main aim of this paper is to give a positive answer. We will give several bounds for the noise intensity matrix under which the solution of the stochastic network will approach a limit set which is in general different from $K_0$.

Throughout this paper we always assume that $\sigma(x)$ is locally Lipschitz continuous and satisfies the linear growth condition. It is therefore known (cf Arnold [1], Friedman [4] or Mao [11, 12]) that given any initial value $x_0 \in R^n$, equation (1.10) has a unique global solution on $t \geqslant 0$ and we denote the solution by $x(t; x_0)$. We will let $|\cdot|$ denote the Euclidean norm in $R^n$. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. If $A$ is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^T A)}$ while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. Moreover, if $A$ is a symmetric matrix, denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ the smallest and largest eigenvalues, respectively.

## 2. Limit sets

The diffusion operator $L$ associated with equation (1.10) is given by

$$L = \sum_{i=1}^n \left( -b_i x_i + \sum_{j=1}^n a_{ij} g_j(x_j) + c_i \right) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n (\sigma\sigma^T(x))_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

where

$$(\sigma\sigma^T(x))_{ij} = \sum_{k=1}^{m} \sigma_{ik}(x)\sigma_{jk}(x).$$

For the $C^2$-function $V$ defined by (1.6) we compute

$$\frac{\partial V(x)}{\partial x_i} = \left( b_i x_i - \sum_{j=1}^{n} a_{ij} g_j(x_j) - c_i \right) \dot{g}_i(x_i)$$

$$\frac{\partial^2 V(x)}{\partial x_i^2} = \left( b_i x_i - \sum_{j=1}^{n} a_{ij} g_j(x_j) - c_i \right) \ddot{g}_i(x_i) + (b_i - a_{ii}\dot{g}_i(x_i))\dot{g}_i(x_i)$$

and

$$\frac{\partial^2 V(x)}{\partial x_i \partial x_j} = -a_{ij}\dot{g}_i(x_i)\dot{g}_j(x_j) \qquad \text{if} \quad i \neq j.$$

Therefore

$$LV(x) = -\sum_{i=1}^{n} \dot{g}_i(x_i) \left( -b_i x_i + \sum_{j=1}^{n} a_{ij} g_j(x_j) + c_i \right)^2 - \frac{1}{2} \sum_{i,j=1}^{n} (\sigma\sigma^T(x))_{ij} a_{ij} \dot{g}_i(x_i) \dot{g}_j(x_j)$$
$$+ \frac{1}{2} \sum_{i=1}^{n} (\sigma\sigma^T(x))_{ii} \left[ \left( b_i x_i - \sum_{j=1}^{n} a_{ij} g_j(x_j) - c_i \right) \ddot{g}_i(x_i) + b_i \dot{g}_i(x_i) \right]. \quad (2.1)$$

In the case when there is no stochastic perturbation, i.e. $\sigma = 0$, we have pointed out in section 1 that $LV \leqslant 0$ and the solution will approach the set $K_0 = \{x \in R^n : LV(x) = 0\}$. The question is: does the stochastic perturbation change this property? It does, of course, for some type of stochastic perturbation, but it may still preserve the property for a certain class of stochastic perturbation. For example, recalling the property that

$$x_i \ddot{g}_i(x_i) \leqslant 0 \qquad x_i \in R \quad 1 \leqslant i \leqslant n$$

and the boundedness of $g_i$ and $\dot{g}_i$, we observe that the sum of the second and third terms on the right-hand side of (2.1) is bounded by $h|\sigma(x)|^2$ for some constant $h > 0$. Hence

$$LV(x) \leqslant -\sum_{i=1}^{n} \dot{g}_i(x_i) \left( -b_i x_i + \sum_{j=1}^{n} a_{ij} g_j(x_j) + c_i \right)^2 + h|\sigma(x)|^2.$$

If $\sigma(x)$ is sufficiently small, for instance

$$|\sigma(x)|^2 \leqslant \frac{1}{h} \sum_{i=1}^{n} \dot{g}_i(x_i) \left( -b_i x_i + \sum_{j=1}^{n} a_{ij} g_j(x_j) + c_i \right)^2$$

we should have $LV(x) \leqslant 0$. In this case, does the solution still approach the set $\{x \in R^n : LV(x) = 0\}$? The following theorem describes the situation.

**Theorem 2.1.** *Assume that $LV(x) \leqslant 0$, namely*

$$-\frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T(x))_{ij}a_{ij}\dot{g}_i(x_i)\dot{g}_j(x_j) + \frac{1}{2}\sum_{i=1}^{n}(\sigma\sigma^T(x))_{ii}\left[\left(b_ix_i - \sum_{j=1}^{n}a_{ij}g_j(x_j) - c_i\right)\ddot{g}_i(x_i) + b_i\dot{g}_i(x_i)\right] \leqslant \sum_{i=1}^{n}\dot{g}_i(x_i)\left(-b_ix_i + \sum_{j=1}^{n}a_{ij}g_j(x_j) + c_i\right)^2 \tag{2.2}$$

*for all $x \in R^n$. Define*

$$K = \{x \in R^n : LV(x) = 0 \quad and \quad H_k(x) = 0, 1 \leqslant k \leqslant m\} \tag{2.3}$$

*where*

$$H_k(x) = \sum_{i=1}^{n}\left(b_ix_i - \sum_{j=1}^{n}a_{ij}g_j(x_j) - c_i\right)\dot{g}_i(x_i)\sigma_{ik}(x). \tag{2.4}$$

*Then*

$$K \neq \emptyset. \tag{2.5}$$

*Moreover, define $d(x; K) = \min\{|x - y| : y \in K\}$, i.e. the distance between $x \in R^n$ and the set K. Then for any initial value $x_0 \in R^n$, the solution $x(t; x_0)$ of equation (1.10) has the property that*

$$\liminf_{t\to\infty} d(x(t; x_0); K) = 0 \quad a.s. \tag{2.6}$$

*that is, almost every sample path of the solution will visit the neighbourhood of K infinitely many times. Furthermore, if for any $x \in K$, there is a neighbourhood $\mathcal{O}_x$ of x in $R^n$ such that*

$$V(y) \neq V(x) \qquad for \quad y \in \mathcal{O}_x \quad y \neq x \tag{2.7}$$

*then for any initial value $x_0 \in R^n$, the solution $x(t; x_0)$ of equation (1.10) has the property that*

$$\lim_{t\to\infty} x(t; x_0) \in K \quad a.s. \tag{2.8}$$

*that is, almost every sample path of the solution will converge to a point in K.*

To prove the theorem let us present two useful lemmas.

**Lemma 2.2.** *Let $A(t)$ and $U(t)$ be two continuous adapted increasing processes on $t \geqslant 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s. Let $\xi$ be a nonnegative $\mathcal{F}_0$-measurable random variable such that $E\xi < \infty$. Define*

$$X(t) = \xi + A(t) - U(t) + M(t) \qquad for \quad t \geqslant 0.$$

*If $X(t)$ is nonnegative, then*

$$\{\lim_{t\to\infty} A(t) < \infty\} \subset \{\lim_{t\to\infty} X(t) < \infty\} \cap \{\lim_{t\to\infty} U(t) < \infty\} \quad a.s.$$

*where $G \subset D$ a.s. means $P(G \cap D^c) = 0$. In particular, if $\lim_{t\to\infty} A(t) < \infty$ a.s., then for almost all $\omega \in \Omega$*

$$\lim_{t\to\infty} X(t, \omega) < \infty \qquad \lim_{t\to\infty} U(t, \omega) < \infty$$

*and*

$$\lim_{t\to\infty} M(t,\omega) \text{ exists and is finite.}$$

This lemma is established by Liptser and Shiryayev [10] (theorem 7 on p 139). Using this we can show the following result.

**Lemma 2.3.** *If (2.2) holds, then for any initial value $x_0 \in R^n$, the solution of equation (1.10) has the properties that*

$$-\mu \leqslant \lim_{t\to\infty} V(x(t;x_0)) < \infty \quad a.s. \tag{2.9}$$

*and*

$$\int_0^\infty \left[ -LV(x(t;x_0)) + \sum_{k=1}^m H_k^2(x(t;x_0)) \right] \mathrm{d}t < \infty \quad a.s. \tag{2.10}$$

*where $H_k(x)$ have been defined by (2.4) above and*

$$\mu = \frac{1}{2}\sum_{i,j=1}^n |a_{ij}| + \sum_{i=1}^n c_i.$$

**Proof.** Fix any initial value and write $x(t;x_0) = x(t)$. By Itô's formula,

$$V(x(t)) = V(x_0) + \int_0^t LV(x(s))\,\mathrm{d}s + M(t)$$

where

$$M(t) = \sum_{i=1}^n \sum_{k=1}^m \int_0^t \left( b_i x_i(s) - \sum_{j=1}^n a_{ij} g_j(x_j(s)) - c_i \right) \dot{g}_i(x_i(s))\sigma_{ik}(x(s))\,\mathrm{d}w_k(s)$$

which is a real-valued continuous local martingale with $M(0) = 0$. It is easy to see from the definition of function $V$ and the properties of functions $g_i$ that

$$V(x) \geqslant -\frac{1}{2}\sum_{i,j=1}^n |a_{ij}| - \sum_{i=1}^n c_i = -\mu.$$

Hence

$$0 \leqslant V(x(t)) + \mu = V(x_0) + \mu - \int_0^t [-LV(x(s))]\,\mathrm{d}s + M(t).$$

An application of lemma 2.2 yields assertion (2.9) as well as

$$\int_0^\infty [-LV(x(s))]\,\mathrm{d}s < \infty \quad \text{a.s.} \tag{2.11}$$

and

$$\lim_{t\to\infty} M(t) \text{ exists and is finite almost surely.} \tag{2.12}$$

To prove the other assertion (2.10), define, for any integer $r \geqslant 1$, a stopping time

$$\tau_r = \inf\{t \geqslant 0 : |M(t)| \geqslant r\}$$

where, and throughout this paper, we set $\inf \emptyset = \infty$. Clearly $\tau_r \uparrow \infty$ a.s. and, by (2.11), $P(\Omega_1) = 1$ where

$$\Omega_1 = \bigcup_{r=1}^\infty \{\omega : \tau_r(\omega) = \infty\}. \tag{2.13}$$

Note from the property of the Ito integral that for any $t \geqslant 0$,

$$E\int_0^{t\wedge\tau_r}\left(\sum_{k=1}^m H_k^2(x(s))\right)\mathrm{d}s = E|M(t\wedge\tau_r)|^2 \leqslant r^2.$$

Letting $t\to\infty$ and using the well-known Fatou lemma, we obtain

$$E\int_0^{\tau_r}\left(\sum_{k=1}^m H_k^2(x(s))\right)\mathrm{d}s \leqslant r^2$$

which yields

$$\int_0^{\tau_r}\left(\sum_{k=1}^m H_k^2(x(s))\right)\mathrm{d}s < \infty \quad \text{a.s.}$$

Therefore there is a subset $\Omega_2$ of $\Omega$ with $P(\Omega_2) = 1$ such that for all $\omega\in\Omega_2$

$$\int_0^{\tau_r(\omega)}\left(\sum_{k=1}^m H_k^2(x(s,\omega))\right)\mathrm{d}s < \infty \qquad \text{for all } r \geqslant 1. \tag{2.14}$$

Now for any $\omega\in\Omega_1\cap\Omega_2$, there is an integer $\bar{r} = \bar{r}(\omega)$, by (2.13), such that $\tau_{\bar{r}}(\omega) = \infty$ and hence by (2.14)

$$\int_0^{\infty}\left(\sum_{k=1}^m H_k^2(x(s,\omega))\right)\mathrm{d}s < \infty.$$

Since $P(\Omega_1\cap\Omega_2) = 1$, we obtain

$$\int_0^{\infty}\left(\sum_{k=1}^m H_k^2(x(s,\omega))\right)\mathrm{d}s < \infty \quad \text{a.s.}$$

This, together with (2.11), implies the required assertion (2.10). The proof is complete. □

We can now begin to prove theorem 2.1.

**Proof of theorem 2.1.** Fix any initial value $x_0$ and write $x(t;x_0) = x(t)$ as before. By lemma 2.3, there is an $\bar{\Omega}\subset\Omega$ with $P(\bar{\Omega}) = 1$ such that for every $\omega\in\bar{\Omega}$,

$$-\mu \leqslant \lim_{t\to\infty} V(x(t,\omega)) < \infty \tag{2.15}$$

and

$$\int_0^{\infty} U(x(t,\omega))\,\mathrm{d}t < \infty \tag{2.16}$$

where

$$U(x) = -LV(x) + \sum_{k=1}^m H_k^2(x) \geqslant 0.$$

But, by the definition of $V$ and property (1.5), we have

$$V(x)\to\infty \text{ if and only if } |x|\to\infty.$$

It therefore follows from (2.15) that

$$\limsup_{t\to\infty} |x(t,\omega)| < \infty.$$

Hence, for every $\omega\in\bar{\Omega}$ there is a positive number $h(\omega)$ such that

$$|x(t,\omega)| \leqslant h(\omega) \qquad \text{for all } t \geqslant 0. \tag{2.17}$$

We now divide the whole proof into three steps.

*Step 1*. We first show that $K \neq \emptyset$. Choose an $\omega \in \bar{\Omega}$. By (2.16),

$$\liminf_{t\to\infty} U(x(t,\omega)) = 0.$$

Hence there is a divergence sequence $\{t_r\}_{r\geqslant 1}$ such that

$$\lim_{r\to\infty} U(x(t_r,\omega)) = 0.$$

By (2.17), $\{x(t_r,\omega)\}_{r\geqslant 1}$ is a bounded sequence so there is a convergence subsequence $\{x(t_{\bar{r}},\omega)\}$ such that

$$\lim_{\bar{r}\to\infty} x(t_{\bar{r}},\omega) = \bar{x} \in R^n.$$

Since $U(\cdot)$ is continuous,

$$U(\bar{x}) = \lim_{\bar{r}\to\infty} U(x(t_{\bar{r}},\omega)) = 0.$$

Noting that $x \in K$ if and only if $U(x) = 0$, we see that $\bar{x} \in K$ so $K$ is nonempty.

*Step 2*. We next claim that

$$\liminf_{t\to\infty} d(x(t,\omega); K) = 0 \qquad \text{for all } \omega \in \bar{\Omega}. \tag{2.18}$$

If this is not true, then for some $\hat{\omega} \in \bar{\Omega}$

$$\liminf_{t\to\infty} d(x(t,\hat{\omega}); K) > 0.$$

So there is a pair of positive numbers $\varepsilon$ and $T$ such that

$$d(x(t,\hat{\omega}); K) > \varepsilon \qquad \text{for all } t \geqslant T.$$

Taking the boundedness of $x(t,\hat{\omega})$ into account, we can find a compact subset $G$ of $R^n$ such that

$$G \cap K = \emptyset \quad \text{and} \quad \{x(t,\hat{\omega}) : t \geqslant T\} \subset G.$$

Since $U(x) > 0$ and is continuous on $x \in G$,

$$\min\{U(x) : x \in G\} \geqslant \bar{\varepsilon} > 0.$$

Then

$$U(x(t,\hat{\omega})) \geqslant \bar{\varepsilon} \qquad \text{for all } t \geqslant T.$$

Consequently,

$$\int_0^\infty U(x(t,\hat{\omega}))\,\mathrm{d}t \geqslant \int_T^\infty U(x(t,\hat{\omega}))\,\mathrm{d}t = \infty$$

which contradicts (2.16), since (2.14) holds for all $\omega \in \bar{\Omega}$ and of course for $\hat{\omega}$. Hence, (2.18) must be true and the required assertion (2.6) follows.

*Step 3*. We finally prove (2.8) under additional condition (2.7). Choose any $\omega \in \bar{\Omega}$. By step 2, there is a divergence sequence $\{t_r\}_{r\geqslant 1}$ such that

$$\lim_{r\to\infty} d(x(t_r,\omega); K) = 0. \tag{2.19}$$

But, due to the boundedness of $\{x(t_r,\omega)\}_{r\geqslant 1}$, there is a convergence subsequence $\{x(t_{\bar{r}},\omega)\}$ such that

$$\lim_{\bar{r}\to\infty} x(t_{\bar{r}},\omega) = \bar{x} \in R^n. \tag{2.20}$$

Combining (2.19) and (2.20) we must have

$$\bar{x} \in K. \tag{2.21}$$

We now claim that

$$\lim_{t\to\infty} x(t,\omega) = \bar{x}. \tag{2.22}$$

If this is false, then

$$\limsup_{t\to\infty} |x(t,\omega) - \bar{x}| > 0. \tag{2.23}$$

By condition (2.7), there is an $\varepsilon > 0$ such that

$$V(y) \neq V(\bar{x}) \qquad \text{for} \quad y \in \bar{S}_\varepsilon(\bar{x}) \quad y \neq \bar{x} \tag{2.24}$$

where $\bar{S}_\varepsilon(\bar{x}) = \{y \in R^n : |y - \bar{x}| \leqslant \varepsilon\}$. Due to the continuity of $x(t,\omega)$ in $t$, we observe from (2.20) and (2.23) that there is an $\bar{\varepsilon} \in (0,\varepsilon)$ and a divergence sequence $\{t_u\}_{u\geqslant 1}$ such that

$$\{x(t_u,\omega) : u \geqslant 1\} \subset \bar{S}_\varepsilon(\bar{x}) - S_{\bar{\varepsilon}}(\bar{x})$$

where $S_{\bar{\varepsilon}}(\bar{x}) = \{y \in R^n : |y - \bar{x}| < \bar{\varepsilon}\}$. Hence, there is a convergence subsequence $\{x(t_{\bar{u}},\omega)\}_{\bar{u}\geqslant 1}$ of $\{x(t_u,\omega)\}_{u\geqslant 1}$ such that

$$\lim_{\bar{u}\to\infty} x(t_{\bar{u}},\omega) = y \in \bar{S}_\varepsilon(\bar{x}) - S_{\bar{\varepsilon}}(\bar{x}). \tag{2.25}$$

Since $V(\cdot)$ is continuous, we derive from (2.22), (2.24) and (2.25) that

$$\lim_{\bar{r}\to\infty} V(x(t_{\bar{r}},\omega)) = V(\bar{x}) \neq V(y) = \lim_{\bar{u}\to\infty} V(x(t_{\bar{u}},\omega)).$$

In other words, $\lim_{t\to\infty} V(x(t,\omega))$ does not exist. But this contradicts (2.15) so (2.22) must hold. Now the required assertion (2.8) follows from (2.21) and (2.22). The proof is therefore complete. □

## 3. Conditions for $LV \leqslant 0$

Theorem 2.1 shows that as long as $LV(x) \leqslant 0$, the nonempty set $K$ exists and the solutions of the neural network under stochastic perturbation will approach this set with probability 1 if the additional condition (2.7) is satisfied. It is therefore useful to know how large a stochastic perturbation the neural network can tolerate without losing the property of $LV(x) \leqslant 0$. Although we pointed out in the previous section that there is some $h > 0$ such that

$$LV(x) \leqslant -\sum_{i=1}^{n} \dot{g}_i(x_i)\left(-b_i x_i + \sum_{j=1}^{n} a_{ij} g_j(x_j) + c_i\right)^2 + h|\sigma(x)|^2$$

we did not estimate the $h$. If we know more precisely about $h$, we can estimate the noise intensity, for instance,

$$|\sigma(x)|^2 \leqslant \frac{1}{h}\sum_{i=1}^{n} \dot{g}_i(x_i)\left(-b_i x_i + \sum_{j=1}^{n} a_{ij} g_j(x_j) + c_i\right)^2$$

to guarantee $LV(x) \leqslant 0$.

In section 1 we have listed the properties of functions $g_i$. Let us now introduce

$$\gamma_i = \max\left\{|\ddot{g}_i(x_i)| : 0 \wedge \left(-1 + \frac{c_i}{b_i}\right) \leqslant x_i \leqslant 1 + \frac{c_i}{b_i}\right\} \qquad 1 \leqslant i \leqslant n. \tag{3.1}$$

The following lemma explains why $\gamma_i$ are defined in the way above.

**Lemma 3.1.** *We always have*

$$\left(b_i x_i - \sum_{j=1}^{n} a_{ij} g_j(x_j) - c_i\right)\ddot{g}_i(x_i) \leqslant (b_i + c_i)\gamma_i \qquad 1 \leqslant i \leqslant n \tag{3.2}$$

*for all $x_i \in R$.*

**Proof.** If $x_i > 1 + c_i/b_i$, $\ddot{g}_i(x_i) < 0$ (due to the property of $\ddot{g}_i$) and, by (1.3), we have

$$b_i x_i - \sum_{j=1}^{n} a_{ij} g_j(x_j) - c_i \geqslant b_i x_i - \sum_{j=1}^{n} |a_{ij}| - c_i = b_i x_i - b_i - c_i > 0$$

so

$$\left(b_i x_i - \sum_{j=1}^{n} a_{ij} g_j(x_j) - c_i\right) \ddot{g}_i(x_i) < 0.$$

If $x_i < 0 \wedge (-1 + c_i/b_i)$, $\ddot{g}_i(x_i) > 0$ and

$$b_i x_i - \sum_{j=1}^{n} a_{ij} g_j(x_j) - c_i \leqslant b_i x_i + \sum_{j=1}^{n} |a_{ij}| - c_i = b_i x_i + b_i - c_i < 0$$

so

$$\left(b_i x_i - \sum_{j=1}^{n} a_{ij} g_j(x_j) - c_i\right) \ddot{g}_i(x_i) < 0.$$

But if $0 \wedge (-1 + c_i/b_i) \leqslant x_i \leqslant 1 + c_i/b_i$, $x_i \ddot{g}_i(x_i) \leqslant 0$ so

$$\left(b_i x_i - \sum_{j=1}^{n} a_{ij} g_j(x_j) - c_i\right) \ddot{g}_i(x_i) \leqslant \left(\sum_{j=1}^{n} |a_{ij}| + c_i\right) |\ddot{g}_i(x_i)| \leqslant (b_i + c_i)\gamma_i.$$

Hence (3.2) always holds. The proof is complete. □

We can now describe a condition for $LV \leqslant 0$.

**Theorem 3.2.** *If*

$$\frac{1}{2}|\sigma(x)|^2 [\max_{1\leqslant i\leqslant n} ((b_i + c_i)\gamma_i + b_i \dot{g}_i(x_i) - \lambda_{\min}(A)|\dot{g}_i(x_i)|^2)] \leqslant \sum_{i=1}^{n} \dot{g}_i(x_i) \left(-b_i x_i + \sum_{j=1}^{n} a_{ij} g_j(x_j) + c_i\right)^2 \quad (3.3)$$

*then* $LV(x) \leqslant 0$.

**Proof.** Compute

$$\begin{aligned}\sum_{i,j=1}^{n} (\sigma\sigma^T(x))_{ij} a_{ij} \dot{g}_i(x_i) \dot{g}_j(x_j) &= \sum_{i,j=1}^{n} \sum_{k=1}^{m} \sigma_{ik}(x)\sigma_{jk}(x) a_{ij} \dot{g}_i(x_i) \dot{g}_j(x_j) \\ &= \sum_{k=1}^{m} \sum_{i,j=1}^{n} \dot{g}_i(x_i)\sigma_{ik}(x) a_{ij} \dot{g}_j(x_j)\sigma_{jk}(x) \\ &\geqslant \sum_{k=1}^{m} \lambda_{\min}(A) \sum_{i=1}^{n} |\dot{g}_i(x_i)\sigma_{ik}(x)|^2 \\ &= \sum_{k=1}^{m} \sum_{i=1}^{n} \lambda_{\min}(A) |\dot{g}_i(x_i)|^2 |\sigma_{ik}(x)|^2. \quad (3.4)\end{aligned}$$

Also, by lemma 3.1,

$$\sum_{i=1}^{n}(\sigma\sigma^T(x))_{ii}\left[\left(b_ix_i-\sum_{j=1}^{n}a_{ij}g_j(x_j)-c_i\right)\ddot{g}_i(x_i)+b_i\dot{g}_i(x_i)\right]$$
$$\leqslant \sum_{i=1}^{n}(\sigma\sigma^T(x))_{ii}[(b_i+c_i)\gamma_i+b_i\dot{g}_i(x_i)]$$
$$= \sum_{k=1}^{m}\sum_{i=1}^{n}[(b_i+c_i)\gamma_i+b_i\dot{g}_i(x_i)]|\sigma_{ik}(s)|^2. \tag{3.5}$$

Substituting (3.4) and (3.5) into (2.1) yields

$$LV(x) = -\sum_{i=1}^{n}\dot{g}_i(x_i)\left(-b_ix_i+\sum_{j=1}^{n}a_{ij}g_j(x_j)+c_i\right)^2-\frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{n}\lambda_{\min}(A)|\dot{g}_i(x_i)|^2|\sigma_{ik}(x)|^2$$
$$+\frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{n}((b_i+c_i)\gamma_i+b_i\dot{g}_i(x_i))|\sigma_{ik}(s)|^2$$
$$= -\sum_{i=1}^{n}\dot{g}_i(x_i)\left(-b_ix_i+\sum_{j=1}^{n}a_{ij}g_j(x_j)+c_i\right)^2$$
$$+\frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{n}((b_i+c_i)\gamma_i+b_i\dot{g}_i(x_i)-\lambda_{\min}(A)|\dot{g}_i(x_i)|^2)|\sigma_{ik}(s)|^2$$
$$\leqslant -\sum_{i=1}^{n}\dot{g}_i(x_i)\left(-b_ix_i+\sum_{j=1}^{n}a_{ij}g_j(x_j)+c_i\right)^2$$
$$+\frac{1}{2}|\sigma(x)|^2[\max_{1\leqslant i\leqslant n}((b_i+c_i)\gamma_i+b_i\dot{g}_i(x_i)-\lambda_{\min}(A)|\dot{g}_i(x_i)|^2)]. \tag{3.6}$$

Using (3.3) we have $LV(x)\leqslant 0$. The proof is complete. □

In the case when $\lambda_{\min}(A)\geqslant 0$ we may use the following easier criterion for $LV(x)\leqslant 0$.

**Corollary 3.3.** *If $A$ is a symmetric nonnegative-definite matrix and*

$$|\sigma(x)|^2\leqslant\frac{2}{h}\sum_{i=1}^{n}\dot{g}_i(x_i)\left(-b_ix_i+\sum_{j=1}^{n}a_{ij}g_j(x_j)+c_i\right)^2 \tag{3.7}$$

*holds for all $x\in R^n$, where*

$$h=\max_{1\leqslant i\leqslant n}[(b_i+c_i)\gamma_i+b_i\beta_i] \tag{3.8}$$

*then $LV(x)\leqslant 0$. (Recall that $\beta_i=\dot{g}_i(0)$ which was defined in section 1.)*

**Proof.** Using the conditions we compute

$$\frac{1}{2}|\sigma(x)|^2[\max_{1\leqslant i\leqslant n}((b_i+c_i)\gamma_i+b_i\dot{g}_i(x_i)-\lambda_{\min}(A)|\dot{g}_i(x_i)|^2)]$$
$$\leqslant \frac{1}{2}|\sigma(x)|^2\max_{1\leqslant i\leqslant n}((b_i+c_i)\gamma_i+b_i\dot{g}_i(x_i)) \leqslant \frac{h}{2}|\sigma(x)|^2$$
$$\leqslant \sum_{i=1}^{n}\dot{g}_i(x_i)\left(-b_ix_i+\sum_{j=1}^{n}a_{ij}g_j(x_j)+c_i\right)^2;$$

that is, (3.3) holds so the conclusion follows from theorem 3.2. □

In the case when $\lambda_{\min}(A) < 0$ we may also have the following easier criterion for $LV(x) \leqslant 0$.

**Corollary 3.4.** *If* $\lambda_{\min}(A) < 0$ *and*

$$|\sigma(x)|^2 \leqslant \frac{2}{\bar{h}}\sum_{i=1}^{n}\dot{g}_i(x_i)\left(-b_ix_i+\sum_{j=1}^{n}a_{ij}g_j(x_j)+c_i\right)^2 \tag{3.9}$$

*holds for all* $x \in R^n$, *where*

$$\bar{h} = \max_{1\leqslant i\leqslant n}\left[\left((b_i+c_i)\gamma_i+b_i\beta_i+|\lambda_{\min}(A)|\beta_i^2\right)\right] \tag{3.10}$$

*then* $LV(x) \leqslant 0$.

**Proof.** Compute, by (3.9) and (3.10),

$$\frac{1}{2}|\sigma(x)|^2[\max_{1\leqslant i\leqslant n}((b_i+c_i)\gamma_i+b_i\dot{g}_i(x_i)-\lambda_{\min}(A)|\dot{g}_i(x_i)|^2)]$$
$$\leqslant \frac{1}{2}|\sigma(x)|^2\left[\max_{1\leqslant i\leqslant n}\left((b_i+c_i)\gamma_i+b_i\beta_i+|\lambda_{\min}(A)|\beta_i^2\right)\right]$$
$$= \frac{\bar{h}}{2}|\sigma(x)|^2 \leqslant \sum_{i=1}^{n}\dot{g}_i(x_i)\left(-b_ix_i+\sum_{j=1}^{n}a_{ij}g_j(x_j)+c_i\right)^2;$$

that is, (3.3) holds so the conclusion follows from theorem 3.2. The proof is complete. □

## 4. An example

In this section we will discuss an example, where we let the number of neurons be two in order to make the calculations relatively easier but the theory of this paper is illustrated clearly. In what follows we will also let $w(\cdot)$ be a one-dimensional Brownian motion.

**Example 4.1.** Consider a two-dimensional stochastic neural network

$$\mathrm{d}x(t) = [-Bx(t)+Ag(x(t))+C]\,\mathrm{d}t+\sigma(x(t))\,\mathrm{d}w(t) \tag{4.1}$$

where

$$B = \begin{bmatrix}3 & 0\\0 & 1.5\end{bmatrix} \qquad A = \begin{bmatrix}2 & 1\\1 & 0.5\end{bmatrix} \qquad C = \begin{bmatrix}1\\2\end{bmatrix}$$

$$g(x) = (g_1(x_1), g_2(x_2))^T \qquad g_1(u) = g_2(u) = \frac{2}{\pi}\arctan(u) \qquad \sigma(x) = (\sigma_1(x), \sigma_2(x))^T$$

and $\sigma(x)$ is locally Lipschitz continuous and bounded. Compute

$$\dot{g}_i(u) = \frac{2}{\pi(1+u^2)} \qquad \text{and} \qquad \ddot{g}_i(u) = -\frac{4u}{\pi(1+u^2)^2}.$$

Clearly,

$$\beta_1 = \beta_2 = \frac{2}{\pi} \qquad \text{and} \qquad \int_0^{\pm\infty} u\dot{g}_i(u)\,\mathrm{d}u = \infty.$$

Moreover, by definition (3.1)

$$\gamma_1 = \gamma_2 = \frac{3\sqrt{3}}{4\pi}$$

since $|\ddot{g}_i(u)|$ reaches the maximum at $u = \pm 1/\sqrt{3}$. Noting that $A$ is nonnegative-definite, we may apply corollary 3.3. Compute by (3.8)

$$h = \max_{i=1,2}[(b_i + c_i)\gamma_i + b_i\beta_i] = \frac{2(2+\sqrt{3})}{\pi}.$$

Therefore, if

$$\sigma_1^2(x) + \sigma_2^2(x) \leqslant \frac{2}{2+\sqrt{3}} \left( \frac{1}{1+x_1^2} \left[ -3x_1 + \frac{4}{\pi}\arctan(x_1) + \frac{2}{\pi}\arctan(x_2) + 1 \right]^2 + \frac{1}{1+x_2^2} \left[ -1.5x_1 + \frac{2}{\pi}\arctan(x_1) + \frac{1}{\pi}\arctan(x_2) + 2 \right]^2 \right) \tag{4.2}$$

then $LV(x) \leqslant 0$. The right-hand side of (4.2) gives a bound for the noise intensity. As long as the noise intensity is smaller than the bound, by theorem 2.1, there is a nonempty set $K$ such that almost every sample path of the solution of equation (4.1) will visit the neighbourhood of $K$ infinitely many times. In particular, if

$$\sigma_1^2(x) + \sigma_2^2(x) \leqslant \frac{1}{2} \left( \frac{1}{1+x_1^2} \left[ -3x_1 + \frac{4}{\pi}\arctan(x_1) + \frac{2}{\pi}\arctan(x_2) + 1 \right]^2 + \frac{1}{1+x_2^2} \left[ -1.5x_1 + \frac{2}{\pi}\arctan(x_1) + \frac{1}{\pi}\arctan(x_2) + 2 \right]^2 \right) \tag{4.3}$$

then we have from (3.6) and the proof of corollary 3.3 that

$$\begin{aligned} LV(x) &\leqslant -\sum_{i=1,2} \dot{g}_i(x_i) \left( -b_i x_i + \sum_{j=1}^{n} a_{ij} g_j(x_j) + c_i \right)^2 + \frac{h}{2}|\sigma(x)|^2 \\ &\leqslant -\frac{2-\sqrt{3}}{2\pi} \left( \frac{1}{1+x_1^2} \left[ -3x_1 + \frac{4}{\pi}\arctan(x_1) + \frac{2}{\pi}\arctan(x_2) + 1 \right]^2 \right. \\ &\quad \left. + \frac{1}{1+x_2^2} \left[ -1.5x_1 + \frac{2}{\pi}\arctan(x_1) + \frac{1}{\pi}\arctan(x_2) + 2 \right]^2 \right). \end{aligned} \tag{4.4}$$

It is therefore easy to see that the set $K$ defined by (2.4) is contained by the following set:

$$K_0 = \left\{ (x_1, x_2)^T \in R^2 : -3x_1 + \frac{4}{\pi}\arctan(x_1) + \frac{2}{\pi}\arctan(x_2) + 1 = 0 \right. \\ \left. \text{and } -1.5x_1 + \frac{2}{\pi}\arctan(x_1) + \frac{1}{\pi}\arctan(x_2) + 2 = 0 \right\}.$$

It is not difficult to show that $K_0 = \{(0.8649, 1.8649)^T\}$, i.e. $K_0$ contains only one point in $R^2$. Since $K$ is nonempty and $K \subseteq K_0$, we must have

$$K = K_0 = \{(0.8649, 1.8649)^T\}.$$

It is not difficult to show that $(0.8649, 1.8649)^T$ is the unique minimum point of function $V(x)$ in this example. We can therefore conclude by theorem 2.1 that all of the solutions of equation (4.1) will tend to $(0.8649, 1.8649)^T$ with probability 1 as long as (4.3) is satisfied. Note that this conclusion is independent of the form of the noise intensity matrix $\sigma(x)$ but only requires that the norm of $\sigma(x)$ be bounded by the right-hand side of (4.3). In other words, we obtain a robustness property of the neural network.

## 5. Further discussions

To close our paper, let us have some further discussions on the way in which noise is introduced into a Hopfield network.

It is known that noise has been introduced into a Hopfield network so that the network can avoid getting trapped into a local minima and hence the time evolution of the network is a motion in state space that seeks out its global minima in the system energy. In such a stochastic Hopfield network, the units are stochastic and the degree is determined by a temperature analogue parameter. The stochastic units are actually introduced to mimic the variable strength with which real neurons fire, delays in synapses and random fluctuations from the release of transmitters in discrete vesicles. By including stochastic units it becomes possible with a simulated annealing technique to try and avoid getting trapped into local minima. By making use of a mean-field approximation the Hopfield network again evolves into a deterministic version, and one can then instead apply mean-field annealing to try and avoid local minima.

In the present paper, the introduced Hopfield network is that with continuous-valued transfer functions, but with added terms corresponding to environmental noise. The noise here is not that which is added into the network on purpose to avoid local minima as mentioned above, but it is the environmental noise which the network cannot avoid. Our contribution here is to present some interesting results on the amount of noise that can be tolerated in a Hopfield neural network while still preserving its limit set or experiencing at least another limit set.

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